Definitions and key facts for section 2.1

Recall a **matrix** A is an $m \times n$ rectangular array of numbers with m rows and n columns.
We label the entry of the ith row and jth column a_{ij} and label the jth column $\mathbf{a}_j$ so that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_j & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix} \qquad \text{where } \mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

The **diagonal entries** of A are $a_{11}, a_{22}, \ldots$ and they form the **main diagonal** of A. A matrix whose non-diagonal entries are zero is called a **diagonal matrix**.
A matrix with all zero entries is a **zero matrix** and is usually denoted by 0.

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose columns are the formed from the corresponding rows of A.

The algebra of matrices:

- two $m \times n$ matrices A and B are **equal**, written $A = B$, if their entries are equal;
- the **sum** two $m \times n$ matrices A and B is the matrix whose entries are the sum of the entries of A and B, written $A + B$;
- and the **scalar multiple** an $m \times n$ matrix A by a scalar c (a real number) is the matrix $c\mathbf{A}$ obtaining by multiplying each entry of A by c.

Basic algebraic properties: Let A, B and C be matrices of the same size and c, d be scalars.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $c(A + B) = cA + cB$
5. $(c + d)A = cA + dA$
6. $c(dA) = (cd)A$

Matrix multiplication
Given a $m \times n$ matrix A and a $n \times p$ matrix B,

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \qquad B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$$

we define the **product** of A and B to be the following $m \times p$ matrix

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

Composition of linear transformations If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^p \to \mathbb{R}^n$ are linear transformations with standard matrices A and B respectively, then

$$T(S(\mathbf{x})) = AB\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^p.$$

Computing AB
We compute the product AB by either of two methods:

- Using the definition, we compute each column of AB as a linear combination of the columns of A using weights from the corresponding column of B.
- Using the **row-column rule**: if $(AB)_{ij}$ is the entry of AB in the ith row and jth column then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Properties of matrix multiplication: Let A be an $m \times n$ matrix, and let B and C have sizes such that the each following expression is defined.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$ for any scalar r
5. $I_m A = A = AI_n$

Warnings for matrix multiplication:
In general, a few familiar properties of multiplication **do not** hold for matrices:

1. In general, $AB \neq BA$.
2. In general, if $AB = AC$, that does not guarantee $B = C$.
3. If $AB = 0$ we cannot conclude in general that either A or B is 0

If A is an $n \times n$ matrix, we define A^k to be the product of k-many copies of A

$$A^k = \underbrace{AA \cdots A}_{k} \text{ and } A^0 = I_n.$$

Properties of the transpose
With A and B of appropriate sizes, we have the following.

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(cA)^T = cA^T$ for any scalar c.
4. $(AB)^T = B^T A^T$.